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LOAD NUMBERS, SOLID EARTH TIDES, AND LIQUID CORE DYNAMICS.(U)
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We examine the elastic deformations of a spheroidal Earth due to static loading conditions at the topmost layer. These deformations are represented by infinite harmonic series having the load numbers as coefficients. We apply the Boussinesq theory for an elastic flat surface to the neighborhood of a loading point on a spheroidal Earth and obtain the asymptotic values for the three sets of load numbers. We evaluate these asymptotic limits for the 1066A Earth model.

We subsequently use the Kummer transformation of infinite series and the newly obtained asymptotic values of the load numbers to obtain expressions representing the: (1) Earth's elastic

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20. ABSTRACT (Continued)

deformation; (2) gravity variation; (3) deflection of the vertical; and (4) strain tensor which is induced by these deformations. The closed form sum of all those harmonic series which are required for such numerical evaluations are obtained and displayed in tabular form.

We finally examine the liquid core/solid mantle interaction by studying the asymptotic expansions of the equations of motion as the frequency of the forcing function tends to zero. We characterize such interaction as being generated by a boundary layer. Such a layer justifies the discontinuity that certain variables must undergo across this interface in order to numerically integrate the Navier-Stokes equations with boundary conditions.

CONTENTS

LOAD NUMBERS, SOLID EARTH TIDES, AND LIQUID CORE DYNAMICS

INTRODUCTION

In NRL Report 8410 [1] we expressed the deformations of an elastic Earth due to static surface loads as infinite series of spherical harmonics. The coefficients of these harmonics were called the load numbers and were obtained by integrating the Navier-Stokes equation throughout the interior of a layered Earth with boundary conditions that had to be satisfied at both the center of the configuration and the loaded upper surface. To numerically evaluate these infinite series, one must: (a) be able to ascertain the load numbers of arbitrarily high order without having to solve each time a boundary problem (this is tantamount to establishing an asymptotic expansion for the sequence of load numbers) and (b) develop practical and efficient ways for numerically summing those series of spherical harmonics once the load numbers are known.

We discuss in this report a question that was left unanswered in our previous publication, that is: why the stress function was assumed to be discontinuous at the core-mantle interface when numerically evaluating the load numbers as a boundary value problem.

This report presents in a systematic and rigorous way procedures suitable for the numerical evaluation of the Earth's spheroidal deformations, which most commonly are called the Earth's tides. For this purpose, we: (a) study the Boussinesq theory for the elastic deformations of a flat plate and show how by applying the Boussinesq solution to the tangent plane at a loading point of the spheroidal Earth we can establish the asymptotic behavior of the load numbers [2]; (b) elaborate on certain numerical procedures which seem to be best suited for the summation of series once the asymptotic behavior of their coefficients has been ascertained; (c) provide in tabular form the closed form expressions of those infinite series of spherical harmonics which are essential for the numerical evaluation of the Earth's tides; and (d) examine the dynamics of a liquid core for a nonrotating Earth to mathematically prove the existence of a boundary layer at the top of the liquid core. This layer provides a justification for the discontinuous behavior of some of the integration variables at the liquid core/solid mantle interface, and furnishes the proper number of free parameters for satisfying the boundary conditions at the loaded surface.

A more detailed discussion of these ideas can be found in a forthcoming publication by Lanzano [3]. In reaching our main results, we want to acknowledge the fact that we have greatly benefited from two older publications by Farrell [4] and Pekeris and Accad [5].

ASYMPTOTIC VALUES OF THE LOAD NUMBERS

The strategy we follow in order to ascertain the asymptotic values of the load numbers can be summarized in the following steps:

(1) One should compare the infinite series expansions which represent the spheroidal deformations and which contain the load numbers as the coefficients of the various harmonics

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with appropriate closed form solutions for the same type of deformations which are valid within a limited region of the spheroid. This is the case in the neighborhood of a loading point where the Boussinesq theory of the flat plate applies and gives rise to a closed form solution.

(2) One should refer both solutions in the region where they overlap to a common reference frame. It is convenient to make use of the cylindrical coordinate system with the loading line as its symmetry axis and the tangent plane to the spheroid at the loading point as the auxiliary plane.

(3) Because the limit of the Legendre polynomials P_n for large values of n is the Bessel function J_0 (of order zero) [6], and similarly the same limit for the derivative $dP_n/d\theta$ is proportional to J_1, one should solve the Boussinesq problem by means of the Hankel transforms of order zero and order one because these functional transforms use those Bessel functions as their kernels.

(4) As a consequence, it follows that the asymptotic values of the three load numbers must be proportional to the Hankel transforms of the two components of the displacement and of the perturbed potential in the Boussinesq problem.

We shall next elaborate statements 1 through 4. To begin with, it is clear that for small angular separations from the loading line, the displacements and the perturbed potential of a layered spheroid must agree with the corresponding quantities referring to the tangent plane at the loading point. In the neighborhood of a loading point, and for the values of density and elastic parameters commonly accepted in the most recent Earth models, the elastic forces dominate the gravitational forces. We can, therefore, not only assume that the density and Lamé parameters remain constant in the neighborhood of a loading point, but also neglect the coupling between the gravitational and elastic forces; this situation enables us to study and separately solve one *elastic* problem and one *gravitational* problem.

In the absence of the gravity field $\vec{g}_0$ of the reference state and of the perturbed gravity $\vec{g}_1$ due to deformation, assuming also: (1) elastic equilibrium, and (2) constancy of the Lamé parameters, the Navier-Stokes equation, which was developed in our previous work, Eqs. (14) and (15) of NRL Report 8410 [1], will simply reduce to

$$(\lambda + 2\mu)\nabla(\nabla \cdot \vec{u}) = \mu\nabla \times \nabla \times \vec{u}, \tag{1}$$

where $\vec{u}$ is here the displacement vector. The above equation should be solved for the material half-space simulating the tangent plane, under the assumption that there is only one normal load concentrated at a given location in the uppermost surface. By the same token, the perturbations in the gravitational potential can be represented by the Poisson equation

$$\nabla^2\phi = -4\pi G\rho_0(\nabla \cdot \vec{u}), \tag{2}$$

with constant density ρ_0, along the material half-space, and by the Laplace equation

$$\nabla^2\phi = 0,$$

in empty space.

Consider a spheroidal Earth bounded by an equipotential surface $r = a$ with a vertical load applied at one of its points Q. We introduce the spherical reference frame $(\vec{OQ}, r, \theta)$ with origin O at the centroid of the Earth and polar axis along the line $\vec{OQ}$; we shall measure the colatitude θ from this axis and assume that the configuration be symmetrical with respect to the line $\vec{OQ}$. This line of application of the load can be taken as the z-axis of a cylindrical reference with origin at the loading point Q, and the tangent plane to the spheroid at Q can be taken as the auxiliary plane upon which the other two variables (r and ψ) will be measured. This cylindrical system shall be denoted by (Q, r, ψ). It is clear that if we take a point T in the neighborhood of Q, the arc (QT) along the equipotential, which is represented by $a\theta$, must be the radial distance in the cylindrical reference system; also that the unit vectors $\hat{e}_z$ and $\hat{e}_r$ of the cylindrical frame must be the limiting positions of the unit vectors $\hat{e}_r$ and $\hat{e}_\theta$, respectively, of the associated spherical frame.

We shall solve both Eqs. (1) and (2) where the displacement vector $\overline{u}$ will have the following representation in cylindrical coordinates:

$$\overline{u} = u(z,r)\hat{e}_z + v(z,r)\hat{e}_r. \tag{3}$$

The spheroidal deformations of a layered spheroid along an equipotential surface $r = a$ due to an applied vertical load of mass m_0 acting at a point Q, and when expressed in spherical coordinates in terms of the load numbers h', l', can be written as follows:

$$\overline{u}(a,\theta) = \frac{am_0}{m} \sum_{n=0}^{\infty} [\hat{e}_r h_n'(a) P_n(\cos\theta) + \hat{e}_\theta l_n'(a) dP_n(\cos\theta)/d\theta]. \tag{4}$$

Here m is the mass of the spheroid within the equipotential surface $r = a$.

Similarly, the potential due to the deformation of the spheroid can be expressed in terms of the third load number k', also in spherical coordinates, as follows:

$$\phi(a,\theta) = -\frac{am_0}{m} \gamma_0(a) \sum_{n=0}^{\infty} k_n'(a) P_n(\cos\theta). \tag{5}$$

Here $\gamma_0(a)$ is the gravitational attraction since we are neglecting the rotational effects.

It is now appropriate to recall that the limit of the Legendre polynomials for large values of the parameter n can be established in terms of the Bessel functions. More specifically, see Ref. 6, p. 155

$$\lim_{n\to\infty} P_n[\cos(\theta/n)] = J_0(\theta). \tag{6}$$

By differentiation, and if use is made of elementary properties of Bessel functions,

$$\lim_{n\to\infty} \frac{1}{n} \frac{dP_n[\cos(\theta/n)]}{d(\theta/n)} = -J_1(\theta). \tag{7}$$

Because of these properties it is convenient to solve Eqs. (1) and (2) by means of the Hankel transforms.

The Hankel transform of order n of a function $f(z, r)$ of two variables with respect to one of its variables is represented by the corresponding capital letter and is defined as

$$F(z, \xi) = \int_0^\infty f(z, r) J_n(\xi r) r\, dr, \tag{8}$$

where J_n is the Bessel function of order n. We can verify that the inversion of Eq. (8) will provide

$$f(z, r) = \int_0^\infty F(z, \xi) J_n(\xi r) \xi\, d\xi. \tag{9}$$

Note that the product ξr of these two variables must be a nondimensional quantity. From Eqs. (8) and (9), it is clear that the dimensions of a set of Hankel transforms $(f;F)$ will differ by the square of the dimension of the ξ or r-variable.

We now revert to Eq. (1) and write the components of the stress as

$$\tau_{zz} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\epsilon_{zz} = (\lambda + 2\mu)\frac{\partial u}{\partial z} + \frac{\lambda}{r}\frac{\partial}{\partial r}(rv);$$

$$\tau_{rz} = 2\mu\epsilon_{rz} = \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right). \tag{10}$$

Here u, v are the components of the displacement vector $\overline{u}$ given by Eq. (3); the notation we use is the one by Sokolnikoff [7].

Consider the system formed by Eqs. (1) and (10); and apply the Hankel transform of order zero to the u and τ_{zz} variables, the H transform of order one to the v and τ_{rz} variables. Denote the H transforms of these four variables by the corresponding capital letters. Then assume that both u and v and their partial derivatives vanish at infinity with $1/r$. Consider the system of equations in the material region of the linear space, which we assume to be the $z \leqslant 0$ half space.

We reach the linear differential system

$$\overline{x}' = A\overline{x}, \tag{11}$$

where the primes denote derivatives with respect to the z-variable and where we have set

$$\overline{x} \equiv \text{Column } (U,\ V,\ T_{zz},\ T_{rz}). \tag{12}$$

The matrix A is

$$A \equiv \begin{bmatrix} 0 & -\lambda\xi/\sigma & 1/\sigma & 0 \\ \xi & 0 & 0 & 1/\mu \\ 0 & 0 & 0 & -\xi \\ 0 & 4\mu\eta\xi^2/\sigma & \lambda\xi/\sigma & 0 \end{bmatrix}, \tag{13}$$

with constant values for $\sigma = \lambda + 2\,\mu$ and $\eta = \lambda + \mu$.

The general solution of Eq. (11) is obtainable by elementary procedures in terms of the eigenvalues β of the A matrix, i.e., in terms of the roots of the characteristic equation

$$\text{Det } (A - \beta I) = 0, \tag{14}$$

where I here stands for the (4×4) identity matrix. The four roots of Eq. (14) are $\beta_1 = \beta_2 = \xi$ and $\beta_3 = \beta_4 = -\xi$.

The eigenvectors or solutions to Eq. (11) can be expressed according to the infinite relationship

$$\overline{x} = \exp\,(\pm\xi z)\, \exp\,[(A \mp \xi I)z]\overline{y} = \exp\,(\pm\xi z)[\overline{y} + z(A \mp \xi I)\overline{y} + \frac{z^2}{2!}(A \mp \xi I)^2\overline{y} + \ldots]. \tag{15}$$

One can obtain two independent solutions,

$$\exp\,(\pm\xi z)\overline{y}_1, \tag{16}$$

by solving the linear system,

$$(A \mp \xi I)\overline{y}_1 = 0. \tag{17}$$

Two additional independent solutions are

$$\exp\,(\pm\xi z)[\overline{y}_2 + z(A \mp \xi I)\overline{y}_2], \tag{18}$$

obtainable by choosing solutions of

$$(A \mp \xi I)^2\overline{y}_2 = 0, \tag{19}$$

which are not common to Eq. (17). In both cases, the infinite expansion appearing in Eq. (15) will terminate.

By following the preceding procedure, we reach the set of four linearly independent solutions:

$$\begin{bmatrix} 1 \\ \mp 1 \\ \pm 2\mu\xi \\ -2\mu\xi \end{bmatrix} \exp(\pm \xi z);$$

$$\begin{bmatrix} \mp \xi z + \mu/\eta \\ \xi z \pm \sigma/\eta \\ -2\mu\xi^2 z \\ \pm 2\mu\xi^2 z + 2\mu\xi \end{bmatrix} \exp(\pm \xi z). \tag{20}$$

Let us now take into account the boundary conditions. The vanishing of the four original variables at $z = -\infty$, means that the transformed variables U, V, T_{zz}, T_{rz} must vanish at minus infinity. If we choose henceforth $\xi > 0$, then the solutions containing $\exp(-\xi z)$ cannot be taken into consideration. To combine the remaining two solutions, we suppose: (a) the transversal component of the stress vanishes at $z = 0$: $\tau_{rz}(0, r) = 0$; and (2) the unit normal load is applied on a spherical cap of radius $r = \alpha$, i.e.,

$$\tau_{zz}(0, r) = -1/\pi\alpha^2 \quad \text{for } r \leqslant \alpha,$$

$$\tau_{zz}(0, r) = 0 \quad \text{for } r > \alpha.$$

It follows then that

$$T_{rz}(0, \xi) = 0,$$

and

$$T_{zz}(0, \xi) = -\frac{1}{\pi\alpha^2} \int_0^\infty J_0(r\xi) r \, dr = -\frac{1}{2\pi} \frac{2J_1(\xi\alpha)}{\xi\alpha},$$

whose limit for a shrinking α is $= -1/2\pi$. Using the previous relations, we finally have

$$U(z, \xi) = -\frac{1}{4\pi\mu\xi} \left(-\xi z + \frac{\sigma}{\eta} \right) \exp(\xi z);$$

$$V(z, \xi) = -\frac{1}{4\pi\mu\xi} \left(\xi z + \frac{\mu}{\eta} \right) \exp(\xi z). \tag{21}$$

We next consider the gravitational problem by solving the Poisson equation within the material tangent plane ($z \leqslant 0$) and the Laplace equation in empty space ($z > 0$).

We introduce the Hankel transform of order zero for the perturbed potential ϕ and denote it by Φ. The Poisson equation then becomes

$$\left(\frac{\partial^2}{\partial z^2} - \xi^2 \right) \Phi(z, \xi) = \frac{2G\rho_0}{\eta} \exp(\xi z), \tag{22}$$

after use has been made of Eq. (21) to express the components U and V of the displacement. The characteristic exponents of Eq. (22) are $\pm\xi$. The solution of the Laplace equation valid for empty space ($z > 0$) must vanish at $z = +\infty$ and must therefore be in the form

$$\Phi_e(z,\xi) = A \exp(-\xi z). \tag{23}$$

The solution of Eq. (22), valid within the elastic medium, can be seen to be:

$$\Phi_i(z,\xi) = \left\{B + \frac{G\rho_0}{\eta\xi} z\right\} \exp(\xi z). \tag{24}$$

However, Φ must be continuous at the interface $z = 0$; this leads to $A = B$. By rewriting the Poisson equation as the divergence of the vector

$$\nabla\phi + 4\pi G\rho_0\overline{u},$$

we realize that the normal component of such vector must be continuous at $z = 0$. In terms of the H transforms, we reach the condition:

$$[\partial\Phi_e/\partial z]_{z=0} = [\partial\Phi_i/\partial z]_{z=0} + 4\pi G\rho_0 U_i(0,\xi); \tag{25}$$

this is so because of the vanishing of $U_e(z,\xi)$. Use of Eq. (25) is instrumental in determining the value of A. We get

$$\begin{aligned} \Phi_e(z,\xi) &= \frac{G\rho_0}{2\mu\xi^2} \exp(-\xi z) \quad \text{for } z \geqslant 0, \text{ and} \\ \Phi_i(z,\xi) &= \frac{G\rho_0}{2\mu\xi^2}\left\{1 + \frac{2\mu\xi z}{\eta}\right\} \exp(\xi z) \quad \text{for } z \leqslant 0. \end{aligned} \tag{26}$$

We are now in the position of comparing two solutions which are both valid in the neighborhood of a loading point and of determining the limit of the three load numbers as the order of the harmonic tends to infinity.

First, by expressing the solutions given by Eqs. (4) and (5) in cylindrical coordinates and by making use of Eqs. (6) and (7), the displacement vector and the perturbed potential for large values of n, may be written:

$$\frac{am_0}{m}\,[\hat{e}_z h_n'(a)J_0(n\theta) - \hat{e}_r n l_n'(a)J_1(n\theta)]; \quad -\frac{am_0}{m}\,\gamma_0(a)k_n'(a)J_0(n\theta). \tag{27}$$

We next represent the Boussinesq solution for $z = 0$ and for small values of the radial distance $r = a\theta$ as integrals of the H transforms according to the following relationships:

$$\begin{aligned} \overline{u}(0,a\theta) &= \hat{e}_z \int_0^\infty U\left\{0,\frac{n}{a}\right\} J_0(n\theta)\frac{n}{a^2}\,dn + \hat{e}_r \int_0^\infty V\left\{0,\frac{n}{a}\right\} J_1(n\theta)\frac{n}{a^2}\,dn; \\ \phi(0,a\theta) &= \int_0^\infty \Phi\left\{0,\frac{n}{a}\right\} J_0(n\theta)\frac{n}{a^2}\,dn. \end{aligned} \tag{28}$$

Note that the integration variable ξ has been relabeled $\xi = n/a$, a legitimate replacement since ξ has the dimension of an inverse length.

Each integral appearing in Eq. (28) can be visualized as an infinite summation; their terms pertaining to large values of n must coincide with the corresponding terms appearing in Eq. (27). Since the Boussinesq solution was written in terms of a unit force whereas the applied load here is $m_0\gamma_0(a)$, we reach the following relationships:

$$\begin{aligned} \frac{am_0}{m} h_n'(a) &= \frac{n}{a^2} U\left\{0,\frac{n}{a}\right\} m_0\gamma_0(a); \\ \frac{am_0}{m} nl_n'(a) &= -\frac{n}{a^2} V\left\{0,\frac{n}{a}\right\} m_0\gamma_0(a); \\ \frac{am_0}{m} \gamma_0(a)k_n'(a) &= -\frac{n}{a^2} \Phi\left\{0,\frac{n}{a}\right\} m_0\gamma_0(a). \end{aligned} \tag{29}$$

By using Eqs. (21) for U and V, and Eqs. (26) for Φ_i, we finally arrive at the asymptotic representation,

$$\begin{Bmatrix} h^* \\ l^* \\ k^* \end{Bmatrix} = \frac{m\gamma_0}{4\pi a^2 \eta} \begin{Bmatrix} -\sigma/\mu \\ 1 \\ -3\rho_0\eta/2\mu\bar{\rho} \end{Bmatrix}. \tag{30}$$

Here $\sigma = \lambda + 2\mu$, $\eta = \lambda + \mu$, $\bar{\rho}$ is the mean density, m is the total mass of the spheroid, and h^*, l^*, k^* are the limits of h_n', nl_n', nk_n' as n approaches infinity. All quantities appearing in Eq. (30) must be evaluated at the outermost surface, $r = a$.

For the 1066A Earth model, these limits can be calculated and are

$$\begin{aligned} h^* &= -11.35767, \\ l^* &= 3.43096, \\ k^* &= -4.70473. \end{aligned} \tag{31}$$

NUMERICAL EVALUATION OF THE GEOPHYSICAL PARAMETERS

Once the asymptotic behavior of the three load numbers has been ascertained, one can numerically evaluate the elements of geophysical interest; these are: (a) the elastic deformations of the surface; (b) the variation of the gravity field in direction and intensity; and (c) the strain tensor induced by the deformations. All these quantities are represented by infinite series of Legendre polynomials, the angular distance θ being measured from the loading line.

Because we know the asymptotic limits h^*, k^*, l^* of h_n', nk_n', nl_n', we can take advantage of the Kummer method for the summation of an infinite series of functions [8,9]. This method consists of subtracting from the given series an appropriately chosen series of known sum whose elements are asymptotically proportional to the series in question. The result can then be expressed as the sum of two infinite series of Legendre polynomials: (a) the first one is multiplied by the known asymptotic value of the load number and its sum can be represented in closed form; (b) the second infinite series has coefficients asymptotically tending to zero, so that it can ultimately be evaluated as a finite sum; the number N of its terms must be so chosen that for values of the subscript n larger than N, the difference between the corresponding load number and its asymptotic value can be considered negligible according to the degree of approximation we plan to achieve.

Let us further examine these ideas. Consider the components u, v of the displacement vector $\vec{u}(a,\theta)$. For the normal component, we get

$$u(a,\theta) = \frac{am_0}{m} \sum_{n=0}^{\infty} h_n'(a)P_n \cong \frac{am_0}{m} \left[h^* \sum_{n=0}^{\infty} P_n + \sum_{n=0}^{N} (h_n' - h^*)P_n \right], \tag{32}$$

the value of N depending on the accuracy to be attained.

In a similar manner, the tangential component v of the displacement can be evaluated according to the scheme:

$$\mathrm{v}(a,\theta) \cong \frac{am_0}{m}\left[l^* \sum_{n=1}^{\infty} \frac{1}{n} \frac{dP_n}{d\theta} + \sum_{n=1}^{N} (nl_n' - l^*) \frac{1}{n} \frac{dP_n}{d\theta}\right]. \tag{33}$$

This is so because l^* is the limit of nl_n' as n goes to infinity. Both infinite series, P_n and $(1/n)\ (dP_n/d\theta)$, that appear in Eqs. (32) and (33) can be summed in closed form; their expressions are given in Table 1.

Table 1 — Closed Form Sum of Certain Harmonic Series of Geophysical Interest

(1) $\sum_{n=0}^{\infty} P_n(\cos\theta) = \frac{1}{2} \operatorname{cosec}\left(\frac{\theta}{2}\right)$

(2) $\sum_{n=0}^{\infty} nP_n(\cos\theta) = -\frac{1}{4} \operatorname{cosec}\left(\frac{\theta}{2}\right)$

(3) $\sum_{n=0}^{\infty} \frac{dP_n}{d\theta} = -\frac{1}{4} \cos\left(\frac{\theta}{2}\right) \operatorname{cosec}^2\left(\frac{\theta}{2}\right)$

(4) $\sum_{n=1}^{\infty} \left(\frac{1}{n} \frac{dP_n}{d\theta}\right) = -\left[\cos\theta + \sin\left(\frac{\theta}{2}\right)\right] \operatorname{cosec}\theta$

(5) $\sum_{n=1}^{\infty} \left(\frac{1}{n} \frac{d^2P_n}{d\theta^2}\right) = \left[1 - \sin^3\left(\frac{\theta}{2}\right)\right] \operatorname{cosec}^2\theta$

(6) $\sum_{n=1}^{\infty} \left[\frac{1}{n} P_n(\cos\theta)\right] = -\ln\left[\sin\left(\frac{\theta}{2}\right)\left(1 + \sin\frac{\theta}{2}\right)\right].$

Next, consider the variation in the gravity field. The potential at a point $Q(a+u,\ \theta)$ of the deformed surface, located at a height u from the point $P(a,\theta)$ on the equipotential $r = a$, can be written:

$$V(Q) = V_0(P) + uV_0'(P) + V_1^*(P) + V_e(P).$$

Here, primes denote derivatives with respect to the normal distance, V_0 is the potential of the original configuration, V_1^* is the potential due to the redistribution of mass, and V_e is the potential of the applied load.

A gravimeter located at Q measures the following acceleration:

$$V'(Q) = V_0'(P) + uV_0''(P) + (V_1^*)'(P) + V_e'(P). \tag{34}$$

We now expand both V_1^* and V_e into spherical harmonics and note that $V_0''(P) = -\gamma_0'(a)$, where

$$\gamma_0(a) = \frac{4\pi G}{a^2} \int_0^a \rho_0(r) r^2 dr. \tag{35}$$

By differentiation, we easily establish that

$$\gamma_0'(a) = -\frac{2}{a}\gamma_0(a) + 4\pi G\rho_0(a). \tag{36}$$

If we neglect the contribution due to the density at P, Eq. (34) can be rewritten as

$$-\gamma(Q) = -\gamma_0(P) + \frac{2}{a}\,u\gamma_0(P) - \frac{n+1}{a}\,V_1^*(P) + \frac{n}{a}\,V_e(P),$$

which leads to

$$\gamma^*(P,Q) = \gamma_0(P) - \gamma(Q) = \frac{m_0}{m}\,\gamma_0(P) \sum_{n=0}^{\infty} [2h_n'(a) - (n+1)k_n'(a) + n]P_n. \tag{37}$$

Finally, the tilt or deviation of the vertical direction can be expressed as the angle between the normal to equipotential surface, $V_0 + V_1^* =$ constant and the geometrical normal to the deformed surface. For simplicity, we shall limit our considerations to the component of the tilt upon the meridian of the spheroid which we denote by $t_1^*(a,\theta)$.

The normal line to the equipotential surface is represented by

$$(1 + k_n')(dP_n/d\theta),$$

whereas the normal to the deformed surface depends on

$$(1/a)(\partial r/\partial\theta),$$

which is proportional to

$$h_n'(dP_n/d\theta).$$

We can then write

$$t_1^*(a,\theta) = \frac{m_0}{m} \sum_{n=0}^{\infty} [1 + k_n'(a) - h_n'(a)]\,\frac{dP_n}{d\theta}. \tag{38}$$

The numerical evaluation of Eqs. (37) and (38) requires the sum of three additional infinite series: nP_n; $(1/n)P_n$; $dP_n/d\theta$. Their sums are also available from Table 1.

We now come to the evaluation of the components of the strain tensor. We use the fundamental Navier-Stokes equation which was expressed as Eq. (39) in Ref. 1 to write:

$$\epsilon_{rr}(a,\theta) = u'(a,\theta) = \sum_{n=0}^{\infty} U_n'(a)P_n =$$

$$= \sum_{n=0}^{\infty}\left[-\frac{2\lambda(a)}{a\sigma(a)}\,U_n(a) + n(n+1)\,\frac{\lambda(a)}{a\sigma(a)}\,V_n(a) + \frac{E_n(a)}{\sigma(a)}\right] P_n\,(\cos\theta).$$

As usual, primes denote derivatives with respect to the radial distance and $\sigma = \lambda + 2\mu$. U, V and E denote the expansions into spherical harmonics of the components u, v, of the displacement vector and of the component τ_{rr} of the stress tensor. We know that $E_n(a)$ is a delta-function with its peak at $\theta =$ 0, so that it will give a zero contribution anywhere else. If we take these facts into account and represent the v component of the displacement vector in terms of the load numbers, we can rewrite the previous equation as

$$\epsilon_{rr}(a,\theta) = -\frac{2\lambda(a)}{a\sigma(a)}\,u(a,\theta) + \frac{m_0}{m}\,\frac{\lambda(a)}{\sigma(a)} \sum_{n=0}^{\infty} n(n+1)l_n'(a)P_n(\cos\theta). \tag{39}$$

We can next verify that

$$\epsilon_{\theta\theta} = \frac{1}{a}\left[u(a,\theta) + \frac{\partial}{\partial\theta}\,\mathrm{v}(a,\theta)\right] = \frac{1}{a}\,u(a,\theta) + \frac{m_0}{m} \sum_{n=0}^{\infty} l_n'(a)\,\frac{d^2P_n}{d\theta^2}. \tag{40}$$

Also

$$\epsilon_{\phi\phi}(a,\theta) = \frac{1}{a}\,[u(a,\theta) + \mathrm{v}(a,\theta)\cot\theta]. \tag{41}$$

The other components of the strain tensor can be shown to vanish when evaluated at the surface $r = a$. The numerical computation of these infinite series which represent the components of the strain tensor necessitates the knowledge of the sum of the series $(1/n)(d^2P_n/d\theta^2)$; t' 's is also given in Table 1.

Table 1 provides the closed form sum of all those series which are required for the numerical evaluation of the geophysical parameters. A detailed derivation of the sum of these series appears in Ref. 3; for some partial results see Ref. 10.

From a computational point of view, few remarks are in order before bringing this section to a close.

(1) The second infinite series that was introduced through the Kummer transformation turns out to be slowly convergent; various numerical algorithms must be used in order to accelerate its convergence; in this regard, the Euler transformation was found to be a useful tool [8,9].

(2) For large values of the order n of the harmonic and for small values of the angle θ measured from the loading line, one can achieve computational economy by approximating $P_n(\cos\theta)$ and $dP_n/d\theta$ by means of J_0 and J_1, as has been described in the previous section.

(3) The functions appearing in Table 1 are not finite at $\theta = 0$ and should not be used in the immediate neighborhood of the loading position. We have remedied to this situation by employing the Boussinesq approximation extended to a suitable neighborhood of the loading point. We resume therefore our considerations on the Boussinesq solution which had reached the representation given by Eq. (21) and try to express it in terms of cylindrical coordinates at the loading point. In other words, we shall invert the Hankel transformation. For this purpose, we use the following integral appearing in Ref. 6:

$$\int_0^\infty \exp(-az)\, J_\nu(bz)\,dz = (C/b)^\nu (a^2 + b^2)^{-1/2},$$

which is valid for $\nu \geqslant 0$, $a \neq 0$, $b \neq 0$ and where $C = (a^2 + b^2)^{1/2} - a > 0$.

Differentiation of the above expression with respect to the parameter a yields

$$\int_0^\infty \exp(-az) J_\nu(bz) z\,dz = [\nu b (C/b)^{\nu-1} + a(1-\nu)(C/b)^\nu](a^2 + b^2)^{-3/2}.$$

The preceding two integral relations for $\nu = 0, 1$ are all that is required to obtain the Boussinesq approximation in terms of cylindrical coordinates (z, r):

$$u(z,r) = -\frac{m_0\gamma_0(a)}{4\pi\mu R}\left[\frac{\sigma}{\eta} + \frac{z^2}{R^2}\right],$$

$$v(z,r) = -\frac{m_0\gamma_0(a)}{4\pi\eta r}\left[1 + \frac{z}{R} + \frac{\eta r^2 z}{\mu R^3}\right]. \tag{42}$$

Here we have set

$$R^2 = r^2 + z^2;\ \ \eta = \lambda + \mu \ \text{ and } \ \sigma = \lambda + 2\mu.$$

LIQUID CORE/SOLID MANTLE DYNAMICS

To justify certain assumptions that were made in carrying out the numerical integration of the Navier-Stokes equations, it is appropriate to examine the physical conditions that might be expected at the interface between liquid core and solid mantle. We shall therefore revert to the linearized version of the Navier-Stokes equations which was obtained in our previous work [1] and establish some fundamental properties of its solutions.

For the scope of a first-order approximation, we can safely ignore the influence of the Earth's rotation; however, we want to study those solutions of the Navier-Stokes equations that represent harmonic oscillations depending on the time factor exp $(i\sigma t)$. We believe that a deeper understanding can be obtained of the conditions existing at the subject interface if we consider the Earth's permanent deformations as the limit of spheroidal oscillations of the harmonic type when their frequency σ tends to zero.

We use here the same notation adopted in the previous report; in particular, recall that the three sets of functions U, V, and R represent the coefficients of the spherical harmonic developments for the components u, v of the displacement vector, and for the potential due to deformation.

Within the liquid core where $\mu = 0$ and if we ignore the effects due to rotation, the equations governing the spheroidal oscillations of constant frequency σ can be obtained from Eqs. (39) of our previous report [1] in a greatly simplified form to be written as follows:

$$\sigma^2 \rho_0 r V + \rho_0 R - \gamma_0 \rho_0 U + \lambda X = 0; \tag{43}$$

$$\sigma^2 \rho_0 U + \rho_0 R' + \gamma_0 \rho_0 X - \rho_0(\gamma_0 U)' + (\lambda X)' = 0; \tag{44}$$

$$r^2 R'' + 2rR' - n(n+1)R = 4\pi G r^2(\rho_0' U + \rho_0 X). \tag{45}$$

For typographical convenience, we have omitted the subscript n referring to the order of the harmonic because our considerations henceforth will apply to any given value of n.

In the preceding equations, primes denote derivatives with respect to the radial distance r, we have also set

$$rX = rU' + 2U - n(n+1)V \tag{46}$$

to represent the dilatation of the material. If we differentiate Eq. (43) with respect to r, subtract from it Eq. (44) and simplify the ensuing result by means of the original Eq. (43), we reach the following:

$$\left(\gamma_0 + \frac{\lambda \rho_0'}{\rho_0^2}\right) X = \sigma^2(V + rV' - U). \tag{47}$$

In the limit, as σ approaches zero while both U, V remain finite, we either reach the adiabatic condition

$$\gamma_0 + \frac{\lambda \rho_0'}{\rho_0^2} = 0, \tag{48}$$

also known as the Adams-Williamson condition, or we must have $X = 0$. This means that the dilatation must vanish.

Pekeris and Accad have discussed the constitution of the liquid core by means of a nondimensional stratification function $\beta(r)$ defined as:

$$\gamma_0 + \frac{\lambda \rho_0'}{\rho_0^2} = \gamma_0 \beta(r). \tag{49}$$

(See Ref. 5). This in turn, is related to the Brunt-Vaisala frequency N^2, according to

$$\beta = -\frac{\lambda}{\rho_0 \gamma_0^2} N^2. \tag{50}$$

Excluding the existence of neutral stratification throughout the whole liquid core, i.e., $\beta \equiv 0$ everywhere, we realize that, as σ approaches zero X must also tend to zero. Solving then Eqs. (43), (46) and (45) under these specific conditions, we find the particular solution

$$U^* = R^*/\gamma_0;\ V^* = \frac{1}{n(n+1)}\,[r(U^*)' + 2U^*];$$

$$r^2(R^*)'' + 2r(R^*)' - [n(n+1) + 4\pi G\rho_0' r^2/\gamma_0]R^* = 0.$$

We must choose that expression for R^* that remains finite at the origin, $r = 0$; this will give rise to only one arbitrary parameter. Furthermore, if we allow that the variable V^* be discontinuous at the core-mantle interface, we will have available only two parameters for the purpose of satisfying the three boundary conditions that are valid at the loading surface, $r = a$.

One plausible physical condition that can justify the use of an extra free parameter is the existence of an infinitesimal boundary layer at the top of the liquid core, because within such layer the dilatation X can switch from zero to a finite value X^*. This arbitrary jump in value can play the role of the additional free parameter.

We shall prove in a rigorous way that such a condition really exists at the core-mantle boundary; we shall accomplish this by considering the asymptotic expansion of the equations of motion with respect to the inverse frequency of oscillation. We shall limit our discussion to a simple physical model consisting of a uniform liquid core surrounded by a uniform solid mantle so that the analytical developments can be simplified without detracting any essential characteristic feature from the phenomenon.

Let us write the fundamental equations of motion for the case of a uniform liquid core, that is when ρ_0 and λ are constant and $\mu = 0$. It is convenient to introduce the constant,

$$A = \frac{4}{3}\pi G\rho_0,$$

which is related to the gravitational acceleration according to

$$\gamma_0(r) = Ar,$$

and use it to define a new nondimensional constant,

$$\alpha = \sigma^2/A,$$

and a new variable $Q = R/A$, which has the dimension of a squared length. In this notation, Eqs. (43) and (45) become:

$$\frac{\lambda}{\rho_0 A}X = r(U - \alpha V) - Q, \tag{51}$$

$$r^2Q'' + 2rQ' - n(n+1)Q = 3r^2X. \tag{52}$$

Equations (46) and (47) can be combined into a three-term relation:

$$rX = \alpha(V + rV' - U) = rU' + 2U - n(n+1)V. \tag{53}$$

We must solve Eqs. (51), (52), and (53) for U, V, and Q. Let us begin with Eq. (53) and equate its second and third member; after some algebraic manipulations, we can write

$$r(U - \alpha V)' + (2 + \alpha)(U - \alpha V) = [n(n+1) - \alpha(\alpha + 1)]V.$$

It is easy to realize that $r^{1+\alpha}$ is an integrating factor for the above equation; this will ultimately lead to its solution in the form

$$U - \alpha V = [n(n+1) - \alpha(\alpha+1)]I(r), \tag{54}$$

where

$$r^{2+\alpha}I(r) = \int_0^r r^{1+\alpha}V(r)\,dr. \tag{55}$$

We next express the rX appearing in the right-hand side of Eq. (52) by means of the second member of Eq. (53), and eliminate the U through Eq. (54). The right-hand side of Eq. (52) can then be written as

$$3\alpha r\{rV' + (1-\alpha)V - [n(n+1) - \alpha(\alpha+1)]I\}.$$

The preceding expression is instrumental in establishing the fact that

$$3\alpha rI(r)$$

is a solution to Eq. (52). This can be verified by the use of the following relations,

$$(rI)' = V - (1+\alpha)I;$$
$$r(rI)'' = rV' - (1+\alpha)V + (1+\alpha)(2+\alpha)I; \tag{56}$$

which are obtainable by differentiating Eq. (55) defining the integral $I(r)$.

The general solution of Eq. (52) can be written as

$$Q = Br^n + 3\alpha rI(r), \tag{57}$$

where B is the only arbitrary constant appearing in our formulation because Q must remain finite at $r = 0$. We shall use Eqs. (54) and (57) to eliminate the integral $I(r)$ between them and express the dilatation X, as given by Eq. (51), in terms of Q alone. This can be used to rewrite Eq. (52) as an equation containing only the unknown function Q. If we adopt the nondimensional independent variable

$$s = r/a,$$

the equation governing the variation of Q becomes

$$s^2 \frac{d^2Q}{ds^2} + 2s\frac{dQ}{ds} - n(n+1)Q + \frac{\rho_0 A a^2}{\lambda\alpha}[\alpha(\alpha+4) - n(n+1)]s^2 Q = Ls^{n+2}. \tag{58}$$

Here L denotes the expression

$$-\frac{\rho_0 A a^2}{\lambda\alpha}[n(n+1) - \alpha(\alpha+1)]Ba^n,$$

which depends on the arbitrary constant B. Equation (58) can be shown to have the particular solution Ds^n, where

$$D = \frac{n(n+1) - \alpha(\alpha+1)}{n(n+1) - \alpha(\alpha+4)}Ba^n.$$

Its general solution is of the form

$$Q = Ds^n + T,$$

where T is the general solution of the equation obtainable from Eq. (58) by deleting its right-hand side. The latter equation can be rearranged into

$$\frac{d^2}{ds^2}(sT) - [C(s) + \nu^2](sT) = 0, \tag{59}$$

where

$$C(s) = \frac{n(n+1)}{s^2} - (\alpha+4)\frac{\rho_0 A a^2}{\lambda},$$

and

$$\nu^2 = \frac{\rho_0 A a^2}{\lambda\alpha} n(n+1) = \frac{\rho_0 A^2 a^2}{\lambda\sigma^2} n(n+1) > 0. \tag{60}$$

When the frequency σ, and consequently the parameter α, decrease to zero and for $s \neq 0$, the values of the function $C(s)$ will become and remain negligible as compared with the constant ν^2. We therefore conclude that for $0 < s \leqslant s_0$, the asymptotic solution of Eq. (59) valid for decreasing values of σ is of the form

$$T = \frac{F}{s} \exp\,[\nu(s - s_0)] + O(\alpha)$$

where F stands for an arbitrary constant and $s_0 = b/a$, b being the radius of the liquid core.

The asymptotic expansion of Q accordingly will be

$$Q = Ds^n + \frac{F}{s} \exp\,[\nu(s - s_0)] + O(\alpha); \tag{61}$$

and the asymptotic expansions of the other pertinent variables U, V, and X can be evaluated therefrom.

In fact, if we express the first of Eqs. (56) in terms of the nondimensional variable s, we get

$$V(s) = s\,\frac{dI}{ds} + (2 + \alpha)I(s).$$

The expression for $I(s)$ is obtainable by equating Eq. (61) to Eq. (57) since they both represent the Q-variable:

$$Q = Ds^n + \frac{F}{s} \exp\,[\nu(s - s_0)] = Ba^n s^n + 3as\alpha I(s).$$

Proceeding along these lines and neglecting the positive powers of the small parameter α (or σ), and since we are considering asymptotic expansions, we find, after elementary operations, that

$$V(s) = \frac{B}{n}(as)^{n-1} + \frac{F\nu}{3a\alpha s} \exp\,[\nu(s - s_0)].$$

Similarly, by using Eqs. (51) and (54), we shall reach the asymptotic representations of U and of the normal stress λX; the results are:

$$U(s) = B(as)^{n-1} + \frac{F}{3a\alpha}\,\frac{n(n+1)}{s^2} \exp\,[\nu(s - s_0)];$$

$$X(s) = \frac{1}{3}\,\frac{\nu^2}{a^2}\,\frac{F}{s} \exp\,[\nu(s - s_0)].$$

If we introduce the nondimensional arbitrary parameters

$$E_1 = Ba^{n-2}, \quad F_1 = F/3\alpha a^2,$$

we can write the previous results in dimensionless form as follows:

$$\begin{aligned} &\frac{1}{a}\,V(s) = \frac{1}{n}\,E_1\,s^{n-1} + \frac{1}{s}\,\nu F_1 \exp\,[\nu(s - s_0)]; \\ &\frac{1}{a}\,U(s) = E_1 s^{n-1} + \frac{n(n+1)}{s^2}\,F_1 \exp\,[\nu(s - s_0)]; \\ &\frac{R(s)}{a\gamma_0(a)} = \frac{R(s)}{a^2 A} = \frac{Q(s)}{a^2} = \left\{1 - \frac{\alpha}{n}\right\} E_1 s^n + \frac{3\alpha F_1}{s} \exp\,[\nu(s - s_0)]; \\ &X(s) = \frac{\alpha\nu^2}{s}\,F_1 \exp\,[\nu(s - s_0)]. \end{aligned} \tag{62}$$

These equations prove that within the liquid core the oscillations induced by a long-period forcing function vary as $\exp\,(1/\sigma)$ and not linearly with σ^2 as one might have originally surmised by inspection of Eqs. (43) and (44).

The monomial terms that appear in the expressions for U and R are regular functions in the interval $0 \leqslant s \leqslant s_0$ and will offset any discontinuity that might arise because of the presence of the exponentials. The V variable varies as $\nu \sim 1/\sigma$ and will tend to increase; we know, however, from other sources that V will be discontinuous at the interface.

On the other hand, the variable X related to both the dilatation and the normal stress in the liquid core depends on the exponential alone since $\alpha\nu^2$ is a finite constant; the X will therefore exhibit a discontinuous behavior typically induced by a boundary layer as described in the following.

The behavior of the exponential term depends on the relative values of $(s - s_0)$ and $\nu \sim 1/\sigma$. For small values of σ and for s much less than s_0, $s - s_0$ is negative, and the exponential of a large negative number provides a negligible contribution; we can say then that, from a practical point of view, the X variable vanishes at large distances from the top layer.

On the other hand, consider the situation where $s - s_0$ is small enough to be comparable in magnitude with the assumed small value of σ; the exponential then provides a sizeable contribution. The smaller the value of σ, the thinner the width of the topmost layer where X is nonvanishing. In the limit as σ approaches zero, the stress distribution can be assumed to behave as a delta-function having a finite spike at $s = s_0$.

For $s = s_0 = b/a$, Eq. (62) become

$$\begin{aligned} \frac{1}{a}\, U(b) &= E_1 s_0^{n-1} + \frac{n(n+1)}{s_0^2} F_1; \\ X(b) &= \frac{\alpha\nu^2}{s_0} F_1 = n(n+1)\, \frac{\rho_0 A a^2}{\lambda s_0} F_1; \\ \frac{R(b)}{a\gamma_0(a)} &= E_1 s_0^n; \end{aligned} \tag{63}$$

whereas V can be taken to be discontinuous at the interface. We have two arbitrary parameters E_1, F_1 plus the value of the discontinuity suffered by V in going from the liquid to the solid layer. These are then three quantities we can use in trying to satisfy the boundary conditions at $r = a$.

This approach was followed in evaluating the load numbers of low order, e.g., of order $n = 2$, 3, 4, which are rather well known from direct physical measurements. The agreement was good. We are therefore convinced that our analytical solution is an adequate representation of the physical conditions existing at the liquid core-solid mantle interface; and believe that the same procedure should be used for the calculation of higher order load numbers which are not known from ground data but which can, nevertheless, play a significant role in the final determination of the spheroidal deformations.

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